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Hilbertian Quantum Theory as the Theory of Complementarity

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It is demonstrated that the notion of complementary physical quantities assumes the possibility of performing ideal first-kind measurements of such quantities. This then leads to an axiomatic reconstruction of the Hilbertian quantum theory based on the complementarity principle and on its connection with the measurement theoretical idealization known as the projection postulate. As the notion of complementary physical quantities does not presuppose the notion of probability, the given axiomatic reconstruction reveals complementarity as an essential reason for the irreducibly probabilistic nature of the quantum theory.

1. INTRODUCTION

Following Dirac (1958) the usual axiomatic reconstructions of the Hilbertian quantum theory take the principle of superposition of states as the basic physical principle on which the theory is to be erected. The essence of that principle is: Any two (distinct) pure states can be superposed to produce a new pure state (distinct from the two others). In this form the principle is, however, rather weak. It only excludes the simplicial shape of the state space and/or the Boolean structure of the proposition system. For further specifications the principle of superposition of states is, more or less explicitly, connected with the measurement theoretical idealization known as the projection postulate. Accompanied with the projection postulate the principle of superposition of states is then (essentially) enough to determine the Hilbertian quantum theory.

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The best-known link between the superposition principle and the projection postulate goes through the so-called covering property. On one hand, the notion of superposition of states may be provided with the symmetry property: If α is a superposition of β and γ, then β is a superposition of γ and α (and γ is a superposition of α and β). This property, when properly formulated within the chosen scheme, appears as the exchange property of the relevant proposition lattice, which (in the atomistic case) is equivalent to the important covering property. On the other hand, the covering property, which always implies the exchange property, is intimately related to the projection postulate. Namely, the existence of a sufficiently rich family of (pure) ideal first-kind measurements (described most naturally in terms of the state-transformations) provides the relevant proposition lattice with that property.[2]

The above link between the superposition principle and the projection postulate is very crucial for the aforementioned axiomatic reconstructions. However, this link contains nothing essentially quantum theoretical. Really, the projection postulate is a general measurement theoretical assumption which does not distinguish between classical and quantum theories. This is true also of the important covering property, which, in particular, holds in any Boolean proposition system. *A fortiori*, the symmetry property of the notion of superposition of states as well as its lattice theoretic counterpart, the exchange property, trivially hold also in the classical case.

The usual axiomatic reconstructions of the Hilbertian quantum theory emphasize thus very strongly the principle of superposition of states as the basic principle of the quantum theory. However, the essence of the traditional quantum theory is contained in the three fundamental quantum principles: the superposition principle, the uncertainty principle, and the complementarity principle. It is known that some natural formulations of these principles are mutually independent (Lahti, 1981a; Lahti and Bugajski, 1981). Moreover, the three principles as such are not enough to determine the Hilbertian quantum theory (Bugajski and Lahti, 1980). Hence, contrary to the usual axiomatic reconstructions, the equal foundational status of these principles should be stressed. In this spirit we shall work out an alternative axiomatic reconstruction of the Hilbertian quantum theory, which is based on the complementarity principle, and on the fact that the notion of complementary physical quantities assumes the possibility of performing ideal first-kind measurements of these quantities.

[2]A detailed and versatile analysis of this link and the resulting axiomatic reconstruction of the Hilbert space quantum mechanics is carried out in Beltrametti and Cassinelli (1981). See also Varadarajan (1968), Bugajska and Bugajski (1973a), and Guz (1981). Another connection between the superposition principle and the projection postulate is proposed in Lahti (1982).

The notion of complementary physical quantities receives its most natural formulation within the so-called operational or convexity approach, which is thus accepted here as the working frame. The basic ingredients of this approach are briefly recalled in Section 2, whereafter the notion of complementary physical quantities is formulated within that approach (Section 3). In Section 4 both intuitive and formal evidence is put forward to support the idea that the notion of complementary physical quantities assumes the possibility of performing ideal first-kind measurements of these quantities. The possibility of performing such measurements on a physical system is guaranteed by the projection postulate, which is formulated in Section 5. The necessary ingredients of the promised axiomatic reconstruction are then put together in Section 6, where it is demonstrated that the state-space of an operational scheme which satisfies the complementarity principle, and hence as a presupposition the projection postulate, can be identified with the Hilbertian one. In this section the Hilbertian quantum theory appears as the theory of complementarity as proposed by Pauli (1980). In the light of this result the probabilistic nature of the quantum theory is reconsidered in Section 7. A formal proof in favor of the old doctrine that the complementarity principle implies the irreducibly probabilistic nature of the quantum theory is given there. In the concluding section the significance of the given axiomatic reconstruction of the Hilbertian quantum theory is discussed. Moreover its consequences on the problem of determining the structure of the so-called Dirac–Heisenberg–Bohr (DHB) quantum theory are pointed out.

2. THE SCHEME[3]

The basic notion of the scheme is *state* of a physical system, and the basic operation is forming *mixtures* of states. The set B of all (normalized) states of the system will thus be equipped with an algebraic structure which allows one to distinguish between the pure and the mixed states of the system. This structure, the underlying structure of the approach, is given in the following axiom.

Axiom. The set of states of a physical system is represented by a norm closed generating cone $\mathbf{V}^+$ for a complete base norm space $(\mathbf{V}, \mathbf{B})$.

[3]It is sufficient to give a brief summary of the operational scheme only. In this we follow most closely Mielnik (1969), Davis and Lewis (1970), and Edwards (1970). For a more extensive discussion of the approach as well as for further references, see Bugajski and Lahti (1981).

$\mathbf{V}$ is partially ordered by the generating cone $\mathbf{V}^+$: for any α, β in $\mathbf{V}$, $\alpha \leqslant \beta$ iff $\beta - \alpha \in \mathbf{V}^+$. As the elements of the cone $\mathbf{V}^+$ are now called states, those of the base $\mathbf{B}$ are called normalized states. The strictly positive linear functional e on $\mathbf{V}$, which, *a posteriori*, serves to define the set of normalized states as $\mathbf{B} = \{\alpha \in \mathbf{V}^+ \colon e(\alpha) = 1\}$, is called the *strength* functional. The fictitious empty state ω, the zero vector of $\mathbf{V}$, and only that, has the strength 0. Because of the convex structure of $\mathbf{B}$, the distinction between *pure states*, extreme elements of $\mathbf{B}$, and *mixed states*, nonextreme elements of $\mathbf{B}$, can now be made. Ex($\mathbf{B}$) denotes the set of pure states in $\mathbf{B}$.

The basic idea of the approach is that any change in the system, like those caused by measurements on it or those associated with its evolutions, can be described through transformations of states of the system. To this end the important notion of *operation* (state-transformation) is introduced. It is assumed that these operations are not too exotic: an operation, when performed on the system, will change a given initial state into a well-defined final state; it does not increase the strength of any state; it acts additively and homogenously on states; it leaves the empty state on its own. Formally, an operation is defined as a positive, norm-nonincreasing, linear mapping ϕ: $\mathbf{V} \to \mathbf{V}$, and the set $\mathbf{O}$ of all formally possible operations on the state space $\mathbf{V}$ is represented as the set of all positive elements in the unit ball of $\mathbf{L}(\mathbf{V})$, the space of bounded linear operators on $\mathbf{V}$ equipped with the strong operator topology.

The set $\mathbf{O}$ is a semigroup with zero 0 and unit I, admitting one thus to perform sequences of operations on states of the system. Further, it is naturally ordered by: for any ϕ_1, ϕ_2 in $\mathbf{O}$, $\phi_1 \leqslant \phi_2$ iff $(\phi_2 - \phi_1)(\alpha) \in \mathbf{V}^+$ for any state $\alpha \in \mathbf{V}^+$. Note also that any operation ϕ in $\mathbf{O}$ with the property $e(\phi\alpha) = e(\alpha)$ for any α in $\mathbf{B}$ is maximal with respect to that order.

Any operation ϕ leads to a detectable *effect* when combined with detecting the strength of a state after it has undergone the operation ϕ. Thus, for any operation ϕ in $\mathbf{O}$ there is associated its detectable effect, denoted as $e \circ \phi$, which is a positive linear functional on $\mathbf{V}$ with $0 \leqslant e \circ \phi \leqslant e$. On the other hand, for any positive linear functional a on $\mathbf{V}$ such that $0 \leqslant a \leqslant e$ there is an operation ϕ_a in $\mathbf{O}$ whose associated effect $e \circ \phi_a$ equals to a: fix any $\beta \in \mathbf{B}$ and define ϕ_a: $\alpha \mapsto \phi_a(\alpha) = a(\alpha)\beta$. Thus, formally, the set of all (formally possible) effects of the physical system is represented by the set $\mathbf{E}$ of elements a in the dual space $(\mathbf{V}^*, e)$ of $(\mathbf{V}, \mathbf{B})$ satisfying $0 \leqslant a \leqslant e$, where the ordering is defined by the dual cone $\mathbf{V}^{+*}$ of $\mathbf{V}^+$.

The set $\mathbf{E}$ of effects is naturally ordered by: for any a, b in $\mathbf{E}$, $a \leqslant b$ iff $(b - a)(\alpha) \geqslant 0$ for any $\alpha \in \mathbf{V}^+$. $(E, \leqslant)$ is a bounded poset with bounds 0 and e. Moreover $\mathbf{E}$ is closed under the mapping $a \mapsto a^\perp := e - a$, which has the properties $(a^\perp)^\perp = a$; if $a \leqslant b$ then $b^\perp \leqslant a^\perp$. In general, however,

$a \mapsto a^{\perp}$ is not an orthocomplementation as $a \wedge a^{\perp} = 0$ does not necessarily hold in $(E, \leqslant)$.

The two more basic notions of the theory are *instruments* and *observables*. An instrument corresponds to an experimental arrangement, defining thus a (regular) family of operations which can be performed on the system with the arrangement. Thus an instrument $\mathcal{I}$ is defined as a map from the Borel sets $B(\mathbb{R})$ of the real line $\mathbb{R}$ into the set of operations which satisfies: (i) $e(\mathcal{I}(\mathbb{R})(v)) = e(v)$ for every $v \in \mathbf{V}$; (ii) for any countable family (E_i) of pairwise disjoint sets in $B(\mathbb{R})$, $\mathcal{I}(\cup E_i) = \Sigma \mathcal{I}(E_i)$ where the sum converges in the strong operator topology. To each instrument there is associated an observable corresponding to the family of the detectable effects of the operations performable with the instrument. Thus an observable $\mathcal{A}$ is defined as an effect-valued measure on the real Borel space $(\mathbb{R}, B(\mathbb{R}))$ with the properties: (i) $\mathcal{A}(\mathbb{R}) = e$; (ii) for any countable family (E_i) of pairwise disjoint sets in $B(\mathbb{R})$ $\mathcal{A}(\cup E_i) = \Sigma \mathcal{A}(E_i)$, where the sum converges in the weak*-topology of $\mathbf{V}^*$.

The above ingredients, the axiom and the definitions thereafter, constitute an *operational* or *convexity* or $(\mathbf{V}, \mathbf{B})$ *scheme* for describing a physical system. Further specifications are needed in order to provide a full description, either classical or quantal, of a physical system within that scheme. In the following we shall meet with three such specifications, the first given by the complementarity principle, the second given by the projection postulate, and the third given by the complementarity principle together with the projection postulate.

3. COMPLEMENTARITY

Complementarity, which embraces probably the most characteristic feature of the quantum theory, is a binary relationship: some A is complementary to some B.[4] To make it concrete, we follow here Pauli (1980) and Bohr (1935) to formulate it as a binary relationship on the set of observables on a physical system. Intuitively, we then say that two *observables are complementary* if the *experimental arrangements* which permit their unambiguous definitions are *mutually exclusive*. Though this conception does not exhaust the general ideas which Bohr developed under his notion of complementarity, it, however, contains an important part of that notion. Moreover, it is just the above explicated part of the notion of complementarity which receives a natural formulation within the chosen scheme so that

[4] Bohr's notion of complementarity has been discussed and analyzed, e.g., in Jammer (1974), Lahti (1980), and Scheibe (1973).

its connection to the projection postulate can be analyzed. Furthermore, this part of the notion of complementarity appears to be sufficient to lead to the axiomatic reconstruction of the Hilbertian quantum theory so that no further components of that notion are required.

In the present approach, any instrument $\mathcal{I}$: $B(\mathbb{R}) \to \mathbf{O}$ uniquely defines an observable $\mathcal{A}$: $B(\mathbb{R}) \to \mathbf{E}$ through the relation $\mathcal{A}(X)(\alpha) = e(\mathcal{I}(X)\alpha)$ for any X in $B(\mathbb{R})$ and α in $\mathbf{B}$. Moreover, each observable is so defined at least by one instrument. Thus the above intuitive definition of the notion of complementarity can be followed rather closely in the present approach. Defining first a relation of mutual exclusiveness of instruments, we then follow the above intuitive idea to define complementary observables. As the proper definitions have already been discovered we only state them here and refer the reader to Lahti and Bugajski (1981) for further argumentation and discussion.

Definition 1. Instruments $\mathcal{I}_1$: $B(\mathbb{R}) \to \mathbf{O}$ and $\mathcal{I}_2$: $B(\mathbb{R}) \to \mathbf{O}$ are *mutually exclusive* iff $\mathcal{I}_1(X) \wedge \mathcal{I}_2(Y) = 0$ for any bounded X and Y in $B(\mathbb{R})$ for which neither $\mathcal{I}_1(X)$ nor $\mathcal{I}_2(Y)$ is maximal.

Definition 2. Observables $\mathcal{A}_1$: $B(\mathbb{R}) \to \mathbf{E}$ and $\mathcal{A}_2$: $B(\mathbb{R}) \to \mathbf{E}$ are *complementary* iff any two instruments $\mathcal{I}_1$ and $\mathcal{I}_2$ uniquely defining these observables are mutually exclusive.

The complementarity principle comprises now the requirement for the existence of complementary observables. Accordingly, we say that an operational description satisfies the *complementarity principle* if there exists at least a pair of (nonconstant) complementary observables.

An immediate consequence of the above two definitions is that two observables $\mathcal{A}_1$ and $\mathcal{A}_2$ are complementary iff $\mathcal{A}_1(X) \wedge \mathcal{A}_2(Y) = 0$ for any bounded X and Y in $B(\mathbb{R})$ for which neither $\mathcal{A}_1(X)$ nor $\mathcal{A}_2(Y)$ equals the unit element e of $\mathbf{E}$. Moreover, if two observables $\mathcal{A}_1$ and $\mathcal{A}_2$ are complementary in the sense of Definition 2 then they are also *probabilistically complementary*: If $\mathcal{A}_1(X)(\alpha) = 1$ for some $\alpha \in \mathbf{B}$, then $\mathcal{A}_2(Y)(\alpha) < 1$, and conversely, for all bounded X and Y in $B(\mathbb{R})$ such that $\mathcal{A}_1(X) \neq e \neq \mathcal{A}_2(Y)$. However, the two notions of complementary observables are not equivalent. Rather, probabilistic complementarity is an essential relaxation of complementarity in the sense of Definition 2. As it is the mutual exclusiveness, given in Definition 2, which prevents any "simultaneous use" of the observables in question, we refer to Definition 2 as the definition of complementary observables in the sense of Pauli (1980) and Bohr (1935). (For a further discussion of these two features of complementarity, see 1981b.)

The two most important features of a $(\mathbf{V}, \mathbf{B})$ description which satisfies the complementarity principle are contained in the following two facts (Lahti and Bugajski, 1981): (1) The result of a sequential measurement of complementary observables depends, in general, on the order in which the observables are measured. (2) There exist at least two effects, say a and b, in $\mathbf{E}$ which are disjoint, i.e., $a \wedge b = 0$, but which are not "orthogonal," i.e., $a \neq e - b$. *A fortiori*, $\mathbf{E}$ is non-boolean.

4. COMPLEMENTARITY AND IDEAL FIRST-KIND MEASUREMENTS

There are both intuitive and formal arguments which support the idea that the notion of complementary physical quantities assumes the possibility of performing ideal first-kind measurements of these quantities. We shall express some of them here.

Calling first for the intuitive arguments we recall that in discussing the notion of complementary physical quantities the advocates of this notion always stress some ideality in that notion. This ideality is usually dressed in expressions like:[5] "an *exact knowledge* of the position...results in the *complete impossibility* of determining the momentum..." (Pauli, 1980), or "the *unambiguous definition* of complementary physical quantities..." (Bohr, 1935), or "complementary properties...*in their pure form*..." (Fock, 1978). To reach this "ideality" mutually exclusive conditions/experimental arrangements must be made use of. To stress it further, it seems to be a basic tenet of the Copenhagen interpretation of the quantum theory that the uncertainty relations provide a kind of "relaxation" of complementarity: giving up the strict ideality the mutual exclusiveness can be avoided, and the possibility for the simultaneous use of the relevant quantities is opened. To witness, we complete the above quotation of Fock:

> Complementary properties reveal themselves in their pure form only in different experiments held in mutually exclusive conditions, whereas in conditions of one and the same experiment they manifest themselves only in an incomplete, modified form (for instance, the incomplete localization in the coordinate and the momentum space permitted by the uncertainty relations).

We turn now to the formal discussion.

To provide the formal evidence for the above idea we shall discuss the most important example of complementary observables, position and

[5] The emphasis in the following quotations is ours.

momentum observables, in the Hilbertian realization of the present approach. The state space is then given as $(T_s(H), T_s(H)_1^+)$, where $T_s(H)_1^+$ denotes the set of all positive normalized elements (statistical operators) of the set $T_s(H)$ of all self-adjoint trace-class operators on the underlying Hilbert space H, equipped with the trace norm. In particular, we recall that in this case the set **E** of effects can be identified with the set $\mathbf{E} = \{A \in L_s(H): 0 \leqslant A \leqslant I\}$ of all self-adjoint operators on H which lie between the null and the unit operators 0 and I.

It is well known that the operations which describe ideal first-kind measurements on the system are exactly of the form ϕ_P: $T_s(H) \to T_s(H)$, $\alpha \mapsto \phi_P \alpha = P\alpha P$ for some projection $P \in P(H)$. The resulting effects of such operations are exactly the extreme-effects, i.e., the projections on H (cf., e.g., Beltrametti and Cassinelli, 1981; Lahti and Bugajski, 1981). On the other hand, each physical quantity which can be represented as a self-adjoint operator on H through its unique spectral measure $B(\mathbb{R}) \to P(H) = \mathrm{Ex}(\mathbf{E})$ admits, in particular, ideal first-kind measurements. Actually, it is the possibility of performing such measurements of an observable, which, *a posteriori*, allows one to represent the observable as a self-adjoint operator.

We shall now consider the canonically conjugate position and momentum observables in the sense of a Schrödinger couple. Thus, without any loss in generality, we consider the canonical free-particle position and momentum observables Q and P in the Hilbert space $H = L_2(\mathbb{R})$. Due to their Fourier equivalence $P = (h/2\pi)F^{-1}QF$, with F denoting the Fourier–Plancherel operator on $L_2(\mathbb{R})$, the spectral measures Q: $B(\mathbb{R}) \to P(H)$, $X \mapsto Q(X)$, and P: $B(\mathbb{R}) \to P(H)$, $Y \mapsto P(Y)$ of position and momentum observables satisfy the operator relation:

$$Q(X) \wedge P(Y) = 0 \qquad \text{for all bounded } X \text{ and } Y \text{ in } B(\mathbb{R}) \tag{1}$$

Hence position and momentum observables Q and P are complementary in the sense of Definition 2. In particular, this indicates that the ideal first-kind measurements of these observables are mutually exclusive, and, *a fortiori* there is no ideal first-kind jointmeasurement of these observables.

Referring now to the notion of fuzzy observable we shall show that even the least deviations from the ideal form of position and momentum observables are enough to break their complementarity in the sense of mutual exclusiveness, and thus to open the possibility of their jointmeasurements. To this end we shall briefly recall the notion of fuzzy observable.

Let $\mathcal{A}$: $B(\mathbb{R}) \to P(H)$, $X \mapsto \mathcal{A}(X)$ be a projection-valued measure (i.e., a standard observable), and f: $\mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$ a probability density function. Any such couple $(\mathcal{A}, f)$ defines, in the weak sense, an effect-valued measure (a generalized or fuzzy or unsharp or approximate observable)

$\mathcal{Q}_f$: $B(\mathbb{R}) \to \mathbf{E}$, $X \mapsto \mathcal{Q}_f(X)$ through the relation

$$\operatorname{tr}\left[\alpha \mathcal{Q}_f(X)\right] = \int_{\mathbb{R}} f(x) \operatorname{tr}\left[\alpha \mathcal{Q}(X+x)\right] dx \qquad \forall \alpha \in T_s(H)_1^+ \tag{2}$$

The function f is taken to describe the unsharpness involved in the measurement or the ambiguity involved in the definition of the observable. It is important that this function could be attached either to the measuring apparatus used to measure the values of the observable (characterizing, e.g., the finite resolution of the measuring apparatus), or to the experimental arrangement used to define the observable (characterizing thus the ambiguities involved in the defining procedure). For a more detailed exposition of this notion we refer to Ali and Emch (1974), Davies (1976), Prugovecki (1976), Ali and Prugovecki (1977).

We shall now apply the notion of fuzzy observable to form new observables from the canonical pair (Q, P). Let Q_n and P_m denote the fuzzy observables defined by the couples $(Q,(n/2)\chi_{[-1/n,1/n]})$ and $(P,(m/2)\chi_{[-1/m,1/m]})$, respectively, where $(n/2)\chi_{[-1/n,1/n]}$ denotes the normalized characteristic function of the symmetric interval $[-1/n, 1/n]$ with $n = 1, 2, \ldots$. With increasing n the functions $(n/2)\chi_{[-1/n,1/n]}$ approach the Dirac delta-function so that $w-\lim_{n\to\infty} Q_n = Q$ and $w-\lim_{m\to\infty} P_m = P$. In Lahti (1981) it is shown that

$$\text{l.b.}\{Q_n(X), P_m(Y)\} \neq \{0\} \quad \text{for all } X \text{ and } Y \text{ in } B(\mathbb{R}) \tag{3}$$

where l.b.{.,.} denotes the set of lower bounds of the relevant effects in $\mathbf{E}$. As the above results holds for any natural numbers n and m we may conclude that even the least deviations from the ideal form of position and momentum observables are enough to destroy the complementarity of these observables in the sense of mutual exclusiveness. However, the fuzzy observables Q_n and P_m are probabilistically complementary so that none of the possible jointmeasurements of these observables which result from (3) is an ideal first-kind measurement (cf. Lahti, 1981). Only in the limit $n \to \infty$ and $m \to \infty$, where the observables Q_n and P_m become again complementary in the sense of Definition 2, the possibility for their ideal first-kind measurements arise.

We conclude that the given formal result together with the above intuitive arguments support the idea that

> The notion of complementary physical quantities assumes the possibility of performing ideal first-kind measurements of these quantities. (4)

We emphasize that this assumption is not of a logical necessity, but it appears to be well justified. In the next section we shall formulate the projection postulate, which in our general scheme guarantees the possibility of performing ideal first-kind measurements on the physical system described by the state space $(\mathbf{V}, \mathbf{B})$.

5. PROJECTION POSTULATE

The possibility of performing ideal first-kind measurements of some observables is formulated as the projection postulate. In the present approach such measurements are most properly described as specially regular operations, called filters, whereas the observables in question are most appropriately characterized through their ranges, i.e., through a suitable subset of effects, called propositions. The projection postulate receives then its formulation as a claim for a one-to-one correspondence between the distinguished class of operations, filters, and the distinguished class of effects, propositions. To formulate this assumption in the present scheme the two intended classes of operations and effects should be excavated.

Let $(\mathbf{V}, \mathbf{B})$ be an operational description. The set $\mathbf{O}_f$ of *filters* is defined as a sufficiently rich family of good operations in $\mathbf{O}$. The qualities sufficiently rich and good, which grasp the qualities pure ideal and first-kind, receive their exact meaning below:

Sufficiency. The set $\mathbf{O}_f \subset \mathbf{O}$ is *sufficiently rich* if

(S1) for any pure state α in $\mathrm{Ex}(\mathbf{B})$ there exists uniquely an operation ϕ_α in $\mathbf{O}_f$ such that $e(\phi_\alpha \beta) = e(\beta)$ implies $\beta = \alpha$ for any β in $\mathrm{Ex}(\mathbf{B})$;

(S2) for any operation ϕ in $\mathbf{O}_f$ there exists an operation ϕ' in $\mathbf{O}_f$ such that the resulting effects $e \circ \phi$ and $e \circ \phi'$ are orthogonal in the sense that $(e \circ \phi)^\perp = e \circ \phi'$.

Purity. An operation ϕ in $\mathbf{O}$ is *pure* if

(P1) $\phi\alpha \in [0,1] \times \mathrm{Ex}(\mathbf{B})$ for any α in $\mathrm{Ex}(\mathbf{B})$.

Ideality. An operation ϕ in $\mathbf{O}$ is *ideal* if

(I1) $e(\phi\alpha) = e(\phi_{\alpha'}\alpha)$ for any α in $\mathrm{Ex}(\mathbf{B})$, with $\alpha' = e(\phi\alpha)^{-1}\phi\alpha$ and $\phi_{\alpha'}$ as in (S1).

First-kindness. An operation ϕ in $\mathbf{O}$ is of the *first-kind* if

(F1) $e(\phi\alpha) = e(\alpha)$ implies $\phi\alpha = \alpha$ for any α in $\mathrm{Ex}(\mathbf{B})$;

(F2) $e(\phi^2\alpha) = e(\phi\alpha)$ for any α in $\mathrm{Ex}(\mathbf{B})$.

The properties (S1) through (F2) which define the set $\mathbf{O}_f$ of filters has already been discussed in Bugajski and Lahti (1980), where also references to some other relevant works can be found. Due to the differences between the present approach and the one employed there, some remarks are, however, called for.

As noted in Bugajski and Lahti (1980), the first sufficiency condition (S1) expresses the common belief that any pure state α can be produced by a particular selection or filtering process ϕ_α, which under the assumptions (F1) and (F2) receives the form $\phi_\alpha\beta = e(\phi_\alpha\beta)\alpha$ for any β in Ex(**B**). In the present scheme for any operation ϕ in **O** there exists an operation ϕ' in **O** such that the effects $e \circ \phi$ and $e \circ \phi'$ resulting from the two operations are orthogonal. With the second sufficiency condition (S2) one guarantees that whenever an effect a results from a "good" operation then also its "negation" $a^\perp$ results from a "good" operation.[6]

The purity (P1) of an operation means simply that it takes a pure state onto a pure state with a possible loss in strength. As a pure state may be interpreted as a maximal-information state (see, e.g., Mielnik, 1969; Beltrametti and Casinelli, 1981), a pure operation leaves the system in a maximal-information state whenever it was in a maximal-information state.

With the so-called ideality assumptions one usually aims at minimalizing the influences on the states caused by an operation performed on the system. In addition to the purity condition (P1) and the first-kindness conditions (F1) and (F2), (I1) aims at that. It claims that an ideal ϕ maps any pure state α onto the closest to α eigenstate of ϕ, disturbing thus the system to a minimal extent.

Of the two first-kindness conditions (F1) and (F2), (F1) claims that if ϕ does not lead to a detectable effect when performed on the system in a pure state α then, provided that ϕ is "good enough," it does not alter the state of the system, either. According to (F2), a repeated application of a "good" operation does not lead to a new effect.

As an immediate consequence of the defining properties of filters, we note that they are not only weakly repeatable [$e(\phi^2\alpha) = e(\phi\alpha)$ for any $\alpha \in$ Ex(**B**)] but also repeatable [$\phi^2\alpha = \phi\alpha$ for any $\alpha \in$ Ex(**B**)], and even idempotent ($\phi^2 = \phi$) provided that any mixed state in **B** can be decomposed into its pure components in Ex(**B**). Moreover, filters satisfy the most usual ideality requirement: if a "good" operation ϕ_1 is performed in the system in a pure state α which is an eigenstate of a "good" operation ϕ_2 [i.e., $e(\phi_2\alpha) = e(\alpha)$] which commutes weakly with ϕ_1 (i.e., $\phi_1 \circ \phi_2$ and $\phi_2 \circ \phi_1$ lead to the same effect), then ϕ_1 leaves the system in a state which is still an eigenstate of ϕ_2.

The set L of *propositions* of an operational description (**V**,**B**) is defined as the set of all extreme-effects a in Ex(**E**) with nonempty certainly-yes-domain $a^1 = \{\alpha \in \mathrm{Ex}(\mathbf{B}) \colon a(\alpha) = 1\}$ together with the null-effect 0:

$$L = \{\alpha \in \mathrm{Ex}(\mathbf{E}) \colon a = 0 \text{ or } a^1 \neq \varnothing\}$$

[6]Ideal first-kind measurements with this property are called perfect measurements by Piron (1976).

Thus propositions are exactly those extreme-effects, which, if they are possible [i.e., $a(\alpha) \neq 0$ for some α in Ex(**B**)], can also be actualized [i.e., there exists an α in Ex(**B**) such that $a(\alpha) = 1$]. As the "fuzzyness" inherent in a (**V**, **B**)-description may be interpreted as resulting from the possible outer disturbances of the system,[7] the restriction to extreme-effects guarantees that a proposition could describe a realizable property of the system.

For a given operational description (**V**, **B**) the set $\mathbf{O}_f$ of filters may or may not exist, and the set **L** of propositions may be trivial $\{0, e\}$. However, for any ϕ in $\mathbf{O}_f$, $\phi \neq 0$, the resulting effect $e \circ \phi$ has a nonempty certainly-yes-domain $(e \circ \phi)^1$, and for any a in **L**, $a \neq 0$, one can associate through the Sasaki-projection-construction (Cassinelli and Beltrametti, 1975) a filter ϕ_a such that the resulting effect $e \circ \phi_a$ equals a. Following Bugajski and Lahti (1980) the projection postulate is now expressed as a requirement for a natural one-to-one correspondence between the distinguished sets $\mathbf{O}_f$ and **L** of filters and propositions.

The projection postulate. An operational description (**V**, **B**) satisfies the *projection postulate* iff

1. the set **O** of operations admits a subset $\mathbf{O}_f$ of filters, and
2. there is a natural one-to-one correspondence Φ between the sets $\mathbf{O}_f$ and **L** with the property $a(\alpha) = e(\Phi(a)\alpha)$ for every $a \in \mathbf{L}$ and $\alpha \in \mathrm{Ex}(\mathbf{B})$.

Though the above projection postulate is very restrictive from the general formal point of view, it is, however, rather plausible from the physical point of view. It guarantees the existence of the important class of operations associated with the pure, ideal, first-kind measurements, but it does not restrict the theory to deal with such measurements only. Thus, in particular, the usual critique against the von Neumann–Lüders projection postulate does not apply here. Really, the major critique against that postulate is not so much against the existence of such measurements described by the postulate but rather against the (apparently erroneous) assumption that such measurements exhaust all the physically relevant measurements.[8]

At the close of this section we point out two important examples of convex descriptions which satisfy the projection postulate.

Consider the state space $(M_{\mathbf{R}}(\Omega), M_{\mathbf{R}}(\Omega)_1^+)$, where the base $M_{\mathbf{R}}(\Omega)_1^+$ consists of all Radon probability measures on a compact phase space Ω. Such a description arises from the general scheme essentially with requiring

[7] In addition to the relevant results of Section 4 see also Davies (1976), Holevo (1973), and Ingarden (1974).

[8] For a review of the rather divergent discussion on the projection postulate see Chap. 11 in Jammer (1974) or Sec. 3.5 in Primas (1981).

the unique decomposability of mixed states (cf., e.g., Bugajski, 1981), and it leads to the classical phase space theory (cf., e.g., Lahti and Bugajski, 1981). In this case one may associate with each Borel subset $X \in B(\Omega)$ of the phase space Ω an operation ϕ_X: $M_{\mathbf{R}}(\Omega) \to M_{\mathbf{R}}(\Omega)$, $\mu \mapsto \phi_X(\mu)$ such that $\phi_X(\mu)(f) := \int_X f\, d\mu$ for any continuous function f: $\Omega \to \mathbb{R}$. The corresponding effect is an extreme effect with a nonempty certainly-yes-domain. One may easily verify that the $(M_{\mathbf{R}}(\Omega), M_{\mathbf{R}}(\Omega)_1^+)$ description satisfies the projection postulate with respect to the couple $(\mathbf{O}_f, \mathbf{L})$, where $\mathbf{O}_f = \{\phi \in \mathbf{O}$: $\phi = \phi_X$ for some $X \in B(\Omega)\}$ and $\mathbf{L} \cong B(\Omega)$.

Consider now the Hilbertian state space $(T_s(H), T_s(H)_1^+)$, where the base consists of all positive normalized trace-class operators on a complex separable Hilbert space H. We shall see that such a description arises from the general scheme essentially with requiring the complementarity principle, and it leads to the usual quantum Hilbert space theory. In this case one may associate with each orthogonal projection $P \in P(H)$ on the Hilbert space H an operation ϕ_P: $T_s(H) \to T_s(H)$, $\alpha \mapsto \phi_P\alpha := P\alpha P$. The corresponding effect is an extreme effect with a nonempty certainly-yes-domain. One may easily verify that the $(T_s(H), T_s(H)_1^+)$ description satisfies the projection postulate with respect to the couple $(\mathbf{O}_f, \mathbf{L})$, where $\mathbf{O}_f = \{\phi \in \mathbf{O}$: $\phi = \phi_P$ for some $P \in P(H)\}$ and $\mathbf{L} \cong P(H)$.

6. THE RECONSTRUCTION

In Section 4 both intuitive and formal evidence was given to support the idea that the notion of complementary physical quantities assumes the possibility of performing ideal first-kind measurements of these quantities. In Section 5 the possibility of performing such measurements on the physical system was formulated as the projection postulate. From now on we accept the view that if an operational scheme satisfies the complementarity principle then without any further physical assumptions it also satisfies the projection postulate. In other words, we propose the following postulate: The projection postulate is an idealizing precondition of the complementarity principle. This assumption has, however, rather strong formal consequences for such a description can (essentially) be identified with a Hilbertian description. As the procedure which leads to this identification is already standard we shall only describe its main steps here.[9]

Let the given $(\mathbf{V}, \mathbf{B})$ description satisfy the complementarity principle, and hence as a presupposition the projection postulate. Let $\mathbf{L}$ be the set of propositions of the $(\mathbf{V}, \mathbf{B})$ description. According to (S2), if a is a proposition then also its negation $a^\perp$ is a proposition, and vice versa. Hence $\mathbf{L}$ is an

[9]The details of this rather lengthy mathematical procedure can be found, e.g., in Beltrametti and Cassinelli (1981).

orthocomplemented poset as a subposet of (Ex(E), $\leqslant$, $^{\perp}$). As demonstrated in Bugajski and Lahti (1980), any proposition of the form $e \circ \phi_\alpha$, $\alpha \in$ Ex(**B**), is an atom of **L**. Moreover, the sufficiency condition (S1) leads to a natural one-to-one correspondence between the sets At(**L**) of atoms of **L** and Ex(**B**) of pure states of **B**: $\alpha \leftrightarrow e \circ \phi_\alpha$. Hence **L** is atomic, and, as a consequence of the complementarity principle, nondistributive. Further, in Bugajski and Lahti (1980) it was demonstrated that the ideality condition (I1) induces on **L** the covering property provided that **L** is a (complete) lattice. The complete lattice property can be gained either by assuming the separability of **L** or by constructing a natural embedding of **L** into a complete orthomodular lattice $\tilde{\mathbf{L}}$ (Bugajska and Bugajski, 1973b). After that the Piron–MacLaren representation theorem can be used to identify **L** with the projection lattice $P(H)$ of a separable Hilbert space H. But as the atoms of **L** and the pure states of **B** stand in a one-to-one correspondence with each other, the pure states of the (**V**, **B**) description can now be identified with the one-dimensional projections on H, i.e., with the extreme elements of the base $T_s(H)_1^+$ for the generating cone $T_s(H)^+$ of the Hilbertian state space $T_s(H)$ on the Hilbert space H. One can now reconstruct the sets **O** and **E** of all operations and of all effects of the description as $\mathbf{O} = \{\phi\colon T_s(H) \to T_s(H)$, ϕ is linear, positive, and contracting$\}$ and, due to the duality $T_s(H)^* \cong L_s(H)$, $\mathbf{E} = \{A \in L_s(H),\ 0 \leqslant A \leqslant \mathrm{I}\}$. In particular, we now have $\mathrm{Ex}(\mathbf{E}) = P(H) = \mathbf{L}$, and $\mathbf{O}_f = \{\phi = \phi_P,\ P \in P(H)\}$, so that the natural correspondence between the sets **L** and $\mathbf{O}_f$ is simply $\Phi(\mathbf{P}) = \phi_P$.

To summarize the above discussion, we have the following result:

> *Theorem.* If a (**V**, **B**) description satisfies the complementarity principle, and hence as a presupposition the projection postulate, then the state space (**V**, **B**) can be identified with a Hilbertian one $(T_s(H), T_s(H)_1^+)$.

7. THE PROBABILISTIC NATURE OF THE QUANTUM THEORY

We shall now apply the above result to provide a formal proof for the old doctrine that the complementarity principle is an essential reason for the irreducibly probabilistic nature of the quantum theory.

In the present approach the description of a physical system is based on its state space (**V**, **B**), and the basic numbers generated by the description are of the form $e(\phi\alpha)$ with ϕ in **O** and α in **B**. They are exactly these numbers which are to be confronted with the actual measurement results. As any operation ϕ in **O** belongs to the range of some instrument $\mathcal{I}$: $B(\mathbb{R}) \to \mathbf{O}$, and as any instrument-(normalized) state pair $(\mathcal{I}, \alpha)$ induces a Kolmogorovian probability space $(\mathbb{R}, B(\mathbb{R}), p(\mathcal{I}, \alpha))$, with the probability measure $p(\mathcal{I}, \alpha)$: $B(\mathbb{R}) \to [0, 1]$, $X \mapsto p(\mathcal{I}, \alpha)(X) := e(\mathcal{I}(X)\alpha)$, these numbers admit a natural *probabilistic interpretation*: For any ϕ in **O** and α in **B**, $e(\phi\alpha) = p(\mathcal{I}, \alpha)(Y)$ is the probability that the measurement of the observable $B(\mathbb{R}) \to \mathbf{E}$, $X \mapsto e \circ \mathcal{I}(X)$ with some of its uniquely defining instruments $\mathcal{I}$: $B(\mathbb{R}) \to \mathbf{O}$ $[\phi \in \mathcal{I}(B(\mathbb{R}))]$ on the physical system in the state α yields a numerical value in the set **Y**.

The general (**V**, **B**) scheme provides thus a *probabilistic scheme* for describing physical systems. However, the probabilistic nature of the scheme is not yet determined. It is open both to reducibly probabilistic and to irreducibly probabilistic cases. Really, in the case $(\mathbf{V}, \mathbf{B}) \cong (M_{\mathbb{R}}(\Omega), M_{\mathbb{R}}(\Omega)_1^+)$ the notion of probability can essentially be eliminated as the pure states, the point measures on Ω, are dispersion free over the extreme effects, i.e., propositions (cf. Bugajski, 1981). On the other hand, in the case $(\mathbf{V}, \mathbf{B}) \cong (T_s(H), T_s(H)_1^+)$ the pure states are not dispersion free, and any state $\alpha \in T_s(H)_1^+$ is a convex combination of at most countably many pure states, so that the notion of probability cannot be eliminated in this case (cf. Beltrametti and Cassinelli, 1981, p. 265).

In Section 5 the projection postulate was formulated as a general measurement theoretical idealization which does not depend on particular classical assumptions or quantum facts. In particular, it is satisfied in classical phase space $(M_{\mathbb{R}}(\Omega), M_{\mathbb{R}}(\Omega)_1^+)$ description as well as in quantum Hilbert space $(T_s(H), T_s(H)_1^+)$ description. This shows also that the projection postulate does not decide whether a (**V**, **B**) description satisfying that postulate is or is not essentially probabilistic. It simply leaves this question open.

In Section 6 we saw that a (**V**, **B**) description which satisfies both the complementarity principle and the projection postulate can be identified with a Hilbertian description $(T_s(H), T_s(H)_1^+)$. Such a description is irreducibly probabilistic. As the projection postulate does not decide on that feature of the theory we come to the following conclusion.

> *Corollary.* The complementarity principle is an essential reason for the irreducibly probabilistic nature of the quantum theory.

It is true that the usual axiomatic reconstructions of the Hilbertian quantum theory reveal, in the very same way, that the superposition principle is an essential reason for the irreducibly probabilistic nature of the quantum theory. However, the present solution has one important advantage over the "canonical solution": The notion of complementary physical quantities does not explicitly, nor implicitly, refer to the notion of probability. Being of nonprobabilistic character the notion of complementary physical quantities does not presuppose the notion of probability. Hence the given solution is satisfactory.

8. DISCUSSION

In the axiomatic reconstruction of the Hilbert-space quantum mechanics from the superposition principle a crucial step is to establish a connection between the superposition principle and the projection postulate. This can be done quite naturally by assuming the symmetry property for the notion of superposition of states. The link is then provided by the covering property of the relevant proposition system, which refers on one hand to the symmetry property of the notion of superposition of states, and on the other hand to the projection postulate.

In the present approach, the axiomatic reconstruction of the Hilbertian quantum theory is also based strongly on a connection between the basic principle, the complementarity principle, and the projection postulate. The connection was reached by arguing that the very notion of complementary physical quantities presupposes the possibility of performing ideal first-kind measurements of these quantities.

In both cases the connection between the first principle and the projection postulate is not a logical necessity, but, rather, the link appears as a well-justified assumption. Thus the two solutions for the axiomatic reconstruction of the Hilbert-space quantum theory share a similar status. They are based on a fundamental quantum principle and on its more or less explicit connection with the measurement theoretical idealization, called the projection postulate.

In the above two cases the fundamental quantum principle receives, from the formal point of view, a rather minor role. Its main function, apart from motivating the structurally important projection postulate, is to exclude the extreme case of a completely reducible proposition system, i.e., to exclude the Boolean proposition structure. This is actually the common feature of any axiomatic reconstruction of the Hilbert-space quantum theory applying the Piron–MacLaren representation theorem, as the properties assumed by that theorem do not distinguish between the classical and the quantum cases. However, it is only in the quantum case where one finds it difficult to justify such assumptions, and, in particular, the crucial projection postulate. As it appears from the first example of Section 5, the projection postulate is a consequence of another measurement theoretical idealization, the classical ideal, which finds its formulation in the requirement for the unique decomposability of mixtures. Thus it could be worth emphasizing that it is only in the quantum (or nonclassical) case where the projection postulate may become an essential building block of the theory. That this is the case in building up the theory either on the superposition principle or on the complementarity principle is clearly indicated in the above two axiomatic reconstructions of the Hilbert-space quantum theory.

In the present approach the projection postulate is rather weak. It guarantees the existence of sufficiently many ideal first-kind measurements, but it does not restrict the theory to such measurements only. As a consequence, the resulting Hilbertian quantum theory is a generalization of the standard quantum mechanics. This appears, in particular, in the fact that the observables are now represented as effect-valued measures and not only as projection-valued measures as is the case in the standard theory. This feature of the present description underlines thus the importance of also other kinds of measurements than those explicated with the projection postulate.

Since Born's proposal for the probability interpretation of the Schrödinger wave function the question of the origin of the notion of probability in quantum theory has continuously been discussed. As the notion of complementary physical quantities does not presuppose the notion of probability, the present axiomatic reconstruction of the Hilbertian quantum theory reveals complementarity as an essential reason for the irreducibly probabilistic nature of the quantum theory. On the other hand, complementarity finds its physical root in the existence of the universal quantum of action, symbolized by the h.

The so-called Dirac–Heisenberg–Bohr quantum theory, which is erected on the three fundamental quantum principles only, has recently been proposed as a candidate for the proper quantum theory (Bugajski and Lahti, 1980). It was shown that such a quantum theory is a generalization of the standard quantum mechanics, but its detailed mathematical structure remained open. From the present viewpoint the Hilbertian quantum theory appears as the theory of complementarity. Thus the DHB-quantum theory assumes now the structure of the convex description based on the Hilbertian state space $(T_s(H), T_s(H)_1^+)$. Moreover, the superposition principle and the uncertainty principle appear then as but two important corollaries of the theory of complementarity.

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